

Record-dependent measures on the symmetric groups

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Abstract

Probability measure P_n on the symmetric group \mathfrak{S}_n is said to be record-dependent if $P_n(\sigma)$ depends only on the set of records of permutation $\sigma \in \mathfrak{S}_n$. A sequence $P = (P_n)_{n \in \mathbb{N}}$ of consistent record-dependent measures determines a random order on \mathbb{N} . In this paper we describe the extreme elements of the convex set of such P . This problem turns out to be related to the study of asymptotic behavior of permutation-valued growth processes, to random extensions of partial orders, and to the measures on the Young-Fibonacci lattice.

Introduction

Let \mathfrak{S}_n be the group of permutations of $[n] := \{1, \dots, n\}$. Position $j \in [n]$ is called a *upper record position* or simply a *record* in permutation $\sigma \in \mathfrak{S}_n$ if $\sigma(j) = \max_{i \in [j]} \sigma(i)$. Let $R(\sigma) \subset [n]$ be the set of records of σ . Probability measure P_n on \mathfrak{S}_n is called *record-dependent* (RD) if P_n is conditionally uniform given the set of records, or, equivalently, if the probability mass function $P_n(\sigma)$ depends only on $R(\sigma)$.

A natural way to connect permutations of different sizes is suggested by viewing the generic permutation $\sigma \in \mathfrak{S}_n$ as a (total) order on $[n]$, in which i precedes j if i appears in a lower position, that is $\sigma^{-1}(i) < \sigma^{-1}(j)$. Restricting the order to smaller set $[n-1]$ yields a projection $\pi_{n-1}^n : \mathfrak{S}_n \rightarrow \mathfrak{S}_{n-1}$ by which permutation σ is mapped to a permutation which we call *coherent* with σ . Likewise, two probability measures, P_n on \mathfrak{S}_n and P_{n-1} on \mathfrak{S}_{n-1} , are coherent

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if the restriction sends P_n to P_{n-1} . It turns out that for coherent measures if P_n is a RD so is P_{n-1} .

In this paper we are interested in coherent sequences of RD-measures $P = (P_n)_{n \in \mathbb{N}}$. A coherent sequence uniquely determines a probability measure, denoted by the same symbol P , on the space of orders on \mathbb{N} . The measure P will be called RD-measure, meaning that the projection of P to each $[n]$ is RD. Our main result (Theorem 2.3 and Proposition 2.5) explicitly characterizes the extreme elements of the convex set of such RD-measures P . The characterization problem belongs to the circle of ‘de Finetti-type’ questions around sufficiency and stochastic symmetries [1, 14], and can be connected to different contexts like the boundary problem for a branching scheme (see e.g. [16]), and processes on causal posets [2, 3].

A straightforward example of RD-measure on \mathfrak{S}_n is

$$P_n(\sigma) = c \theta^{|R(\sigma)|}, \quad \sigma \in \mathfrak{S}_n, \quad (1)$$

where $\theta \in (0, \infty)$ and $c = c(n, \theta)$ is a normalizing constant. By the “fundamental bijection” on \mathfrak{S}_n this measure is mapped to the Ewens distribution on permutations [19, 7]. However, unless $\theta = 1$ these measures are not coherent in the sense of the present paper. A qualitative difference appears if we look how the number of records $|R(\sigma)|$ grows with n . For (1) the order of growth is logarithmic, while for the measures studied here the right scale for $|R(\sigma)|$ is linear, with the case of uniform P_n ’s being the only exception.

The case $\theta = 1$, which is the uniform distribution, is special among the RD measures. The uniform distribution on \mathfrak{S}_n appears from ranking data from continuous distributions, and in this connection various statistics related to the record samples, such as record times, record values, interrecord times and others attracted lots of attention. We refer the reader to [4] for a review of both classical and more recent results on the theory of records.

In the literature there is a number of other permutation growth models that have a similar pattern: the probability mass function depends, for each n , on some statistic S , and coherence of random permutations of different sizes is defined via a system of projections. For S partition into cycles the model can be embedded in the theory of exchangeable partitions [16, 25]. For S the set of descents, a coherent sequence of random permutations is associated with a random order on \mathbb{N} which has the property of spreadability [9]. The partially exchangeable partitions [24] and random sequences of compositions of integers [17, 7] can be also recast in terms of coherent random permutations. Furthermore, certain parametric deformations of the uniform distribution fit in the framework, with S being a numerical statistic like e.g. the number of cycles, descents, pikes, inversions etc (see [10, 7] for examples of the kind and references).

Both the choice of statistic S and the choice of system of projections connecting symmetric groups affect properties of the permutation model. If we replace $|R(\sigma)|$ in (1) by the number of *lower* records the measures will be coherent under our projections π_{n-1}^n . On the other hand, changing the system of projections one can achieve coherence of (1). The alternative projections were considered in [7] for permutations with distribution depending on statistics of both upper and lower records. Comparing with the previous work, the main distinction of the present setting is in the structure of the set of extreme RD-measures P . In e.g. [16, 7, 17, 24] the extremes are described in terms of infinitely many continuous parameters. In contrast to that, in our model the parametrization of the extreme RD-measures involves an integer sequence and a real number.

Somehow unexpectedly such parametrization already appeared in the literature in the work of Goodman and Kerov [12]. They studied the *Young-Fibonacci* lattice, which is an important example of a *differential poset* introduced by Stanley [26] and Fomin [6]. Goodman and Kerov identified the Martin boundary of the Young-Fibonacci lattice which turned to be very similar to the result of our Theorem 2.3 (see Section 6 for more details). Although we see some further similarities, the conceptual reason for the relation between the RD-measures and the Young-Fibonacci lattice remains unclear and it would be very interesting to have more clarity in this point.

The prototypical instance of a differential poset is the *Young graph*. The study of *coherent* measures on the Young graph and its generalizations is a very deep subject related, in particular, to the theory of group representations, total positivity of matrices, Schur symmetric functions and their generalizations like the Macdonald polynomials, see [16] and references therein. In particular, the identification of the extreme coherent measures (which can be viewed as a certain analogue of our Theorem 2.3) is equivalent to the classification of the characters and finite factor representations of the infinite symmetric group $S(\infty)$. Our results on RD-measures concern a particular Bratteli diagram thus, in principle, they can be also interpreted along the lines of the representation theory of the so-called *AF*-algebras, see [16, Introduction] and [27].

Finally, we mention that some of the extreme RD-measures can be viewed as *order-invariant measures on fixed causal sets*, as introduced by Brightwell and Luczak [3]. We add details to this aspect of our study in Section 6.

1 Record-dependent measures and orders on \mathbb{N}

We write permutation $\sigma \in \mathfrak{S}_n$ in the one-row notation as a word $\sigma(1) \dots \sigma(n)$. Note that position 1 is the minimal and $\sigma^{-1}(n)$ is the maximal element of the set of records $R(\sigma)$ defined in Introduction. For instance, permutation $\sigma = 265714 \in \mathfrak{S}_7$ has $R(\sigma) = \{1, 2, 4\}$.

With $\sigma \in \mathfrak{S}_n$ one associates the order on $[n]$ in which letter i precedes j if i appears in a lower position, meaning that $\sigma^{-1}(i) < \sigma^{-1}(j)$. The restriction of the order to $[n-1]$ yields a projection $\pi_{n-1}^n : \mathfrak{S}_n \rightarrow \mathfrak{S}_{n-1}$ by which letter n is removed from the permutation word: for $i \in [n-1]$

$$\pi_{n-1}^n(\sigma)(i) = \begin{cases} \sigma(i), & \text{for } i < \sigma^{-1}(n), \\ \sigma(i+1) - 1, & \text{for } i \geq \sigma^{-1}(n), \end{cases}$$

For example, 3412 is mapped to 312. More generally, projection $\pi_m^n : \mathfrak{S}_n \rightarrow \mathfrak{S}_m$ is defined for $1 \leq m < n$ as the operation of deleting letters $m+1, \dots, n$ from the permutation word $\sigma(1) \dots \sigma(n)$.

Let \mathcal{O} be the projective limit of the symmetric groups \mathfrak{S}_n taken together with projections π_{n-1}^n . Thus \mathcal{O} consists of coherent sequences $(\sigma_n)_{n \in \mathbb{N}}$, which have $\sigma_n \in \mathfrak{S}_n$ and $\sigma_{n-1} = \pi_{n-1}^n(\sigma_n)$ for $n > 1$. Let π_n^∞ denote the coordinate map sending $(\sigma_1, \sigma_2, \dots) \in \mathcal{O}$ to $\sigma_n \in \mathfrak{S}_n$. The coherence of permutations immediately implies:

Proposition 1.1. *\mathcal{O} is in bijection with the set of total orders on \mathbb{N} . The projection π_n^∞ amounts to restricting the order from \mathbb{N} to $[n]$.*

In what follows we identify elements of \mathcal{O} with the orders corresponding to them. We endow each \mathfrak{S}_n with the discrete topology, and endow \mathcal{O} with the topology of projective limit, which corresponds to the coordinate-wise convergence. In this topology \mathcal{O} is a compact totally disconnected space.

Let $\mathcal{M}(\mathcal{O})$ be the space of Borel probability measures on \mathcal{O} . Each measure $P \in \mathcal{O}$ has marginal measures $P_n := \hat{\pi}_n^\infty(P)$ which satisfy the coherence condition $P_{n-1} = \hat{\pi}_{n-1}^n(P_n)$ for $n > 1$, where and henceforth \hat{f} denotes the pushforward of measures under mapping f . Conversely, by Kolmogorov's measure extension theorem each coherent sequence (P_n) determines a unique measure on \mathcal{O} .

Let $\mathcal{M}_R(\mathfrak{S}_n)$ be the set of RD-measures on \mathfrak{S}_n , as defined in Introduction. We call $P \in \mathcal{M}(\mathcal{O})$ RD-measure if the marginal measures satisfy $P_n \in \mathcal{M}_R(\mathfrak{S}_n)$ for every n . We denote $\mathcal{M}_R(\mathcal{O})$ the set of such RD-measures.

Lemma 1.2. *The permutation statistic $R(\sigma)$ is consistent with projections $\hat{\pi}_{n-1}^n$: if P_n is RD-measure on \mathfrak{S}_n then its projection $P_{n-1} := \hat{\pi}_{n-1}^n(P_n)$ is RD-measure on \mathfrak{S}_{n-1} .*

Proof. Recall that for $\sigma \in \mathfrak{S}_n$ position $\sigma^{-1}(n)$ is the maximal element of $R(\sigma)$. Note that σ can be uniquely recovered from $\sigma^{-1}(n)$ and $\pi_{n-1}^n(\sigma)$. Thus, for $\tau \in \mathfrak{S}_{n-1}$ and $A \subset [n]$ the number of permutations σ which satisfy $R(\sigma) = A$ and $\tau = \pi_{n-1}^n(\sigma)$ depends only on A and $B := R(\tau) \subset [n-1]$. Specifically, this number is one if for some $j \in [n]$ the set A is obtainable by the deletion-insertion operation: delete from B all elements not smaller than j then insert j in the remaining set; or otherwise the number is zero. For instance, taking $n = 7$ and $B = \{1, 3, 5\}$ the deletion-insertion operation yields that the above number is one for

$$A = \{1\}, \{1, 2\}, \{1, 3\}, \{1, 3\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 3, 5, 6\}, \{1, 3, 5, 7\}.$$

The assertion follows since

$$P_{n-1}(\tau) = \sum_{A \subset [n]} \sum_{\sigma: \pi_{n-1}^n(\sigma) = \tau, R(\sigma) = A} P_n(\sigma).$$

□

An alternative coordinatization of permutations is sometimes useful. For permutation σ define r_i as the rank of $\sigma(i)$ among $\sigma(1), \dots, \sigma(i)$. That is to say, $r_i = k$ if $\sigma(i)$ is the k th minimum in $\{\sigma(1), \dots, \sigma(i)\}$. We shall call the r_i 's *ranks* (other terminology used in the literature is ‘relative ranks’ or ‘initial ranks’). The correspondence $\sigma \mapsto (r_1, \dots, r_n)$ is a bijection between \mathfrak{S}_n and $[1] \times \dots \times [n]$. Under the uniform distribution on \mathfrak{S}_n the ranks are independent random variables, with r_i uniformly distributed on $[i]$.

Remark. The representation of permutations by the rank sequences does not sit well with the projections π_{n-1}^n , which in terms of the ranks r_i are rather involved. On the other hand, projections $(r_1, \dots, r_n) \mapsto (r_1, \dots, r_{n-1})$ are useful to study other classes of measures on permutations, e.g. (1).

The set of RD-measures $\mathcal{M}_R(\mathfrak{S}_n)$ is a simplex with 2^{n-1} extreme elements P^ρ : these are the measures obtained by conditioning the uniformly distributed permutation on the event $R(\sigma) = \rho$ for given set of records $\rho \subset [n]$ satisfying $1 \in \rho$. In the rank coordinates P^ρ is a product measure: the r_i 's are independent, r_i is uniformly distributed on $[i-1]$ for $i \notin \rho$, and $r_i = i$ almost surely for $i \in \rho$.

The convex set $\mathcal{M}_R(\mathcal{O})$ is a projective limit of the finite-dimensional simplices $\mathcal{M}_R(\mathfrak{S}_n)$. By the general theory (see e.g. [11]) $\mathcal{M}_R(\mathcal{O})$ is a Choquet simplex, i.e. a convex compact set with the property of uniqueness of representation of the generic point as a convex mixture over the set of extreme elements $\text{ext } \mathcal{M}_R(\mathcal{O})$. In view of the property it is important to determine the extremes.

2 Constructions of the extreme RD-measures

Let $P^* \in \mathcal{M}_R(\mathcal{O})$ be the measure whose projection to \mathfrak{S}_n is the uniform distribution for every n . A characteristic feature of the random order on \mathbb{N} with distribution P^* is exchangeability, that is invariance of the distribution under bijections of \mathbb{N} . The order can be neatly constructed in terms of a sequence (ξ_i) of independent random variables uniformly distributed on the unit interval, by letting i to precede j iff $\xi_i < \xi_j$. It is clear from this construction that the exchangeable order on \mathbb{N} is almost surely dense and do not have either maximal, or minimal element. Thus, \mathbb{N} with this order is isomorphic as an ordered space to $(\mathbb{Q}, <)$ P^* -almost surely, as is clear from a classical characterization going back to Hausdorff [13, Section III.11].

Another well-known fact about the uniform distribution on \mathfrak{S}_n is that the number of records in a random permutation sampled from this distribution satisfies $|R(\sigma)|/\log n \rightarrow 1$ in probability (see e.g. [4]). It will be clear from what follows that P^* is the only RD-measure with sublinear growth of $|R(\sigma)|$ as $n \rightarrow \infty$. We shall introduce next a family of random orders for which the number of records is asymptotic to n . The idea is to exploit the ranks as in the construction of extreme elements of $\mathcal{M}_R(\mathfrak{S}_n)$. To that end, we need some preliminaries.

Let \mathfrak{S} be the set of bijections (also called infinite permutations) $\sigma : \mathbb{N} \rightarrow \mathbb{N}$. Bijection σ defines an order on \mathbb{N} by the familiar rule: i precedes j iff $\sigma^{-1}(i) < \sigma^{-1}(j)$. This order is of the type of $(\mathbb{N}, <)$, with j th minimal element being $\sigma(j)$. By the virtue of this correspondence we may view \mathfrak{S} as a subset of \mathcal{O} . Note that the support of the exchangeable order is disjoint with \mathfrak{S} , i.e. $P^*(\mathfrak{S}) = 0$.

With $\sigma \in \mathfrak{S}$ we associate an infinite sequence of ranks r_1, r_2, \dots , defining r_i as the rank of $\sigma(i)$ among $\{\sigma(1), \dots, \sigma(i)\}$. More generally, for $i \leq j$ let $r_{i,j}$ be the rank of $\sigma(i)$ among $\{\sigma(1), \dots, \sigma(j)\}$. The bivariate array is determined by the diagonal entries (r_i) by the virtue of the recursion

$$r_{i,i} = r_i, \quad r_{i,j+1} = r_{i,j} + 1(r_{j+1} \leq r_{i,j}), \quad (2)$$

where $1(\dots)$ is 1 when the condition \dots is true and is 0 otherwise. Moreover, the sequence $r_{i,i}, r_{i,i+1}, \dots$ is nondecreasing and eventually stabilizes at the value

$$\sigma(i) = \lim_{j \rightarrow \infty} r_{i,j}. \quad (3)$$

Thus $\sigma \in \mathfrak{S}$ is uniquely determined by the sequence of ranks (r_i) . This correspondence suggests a criterion to identify the sequences of ranks corresponding to infinite permutations.

Lemma 2.1. *A sequence $(r_i)_{i \in \mathbb{N}}$ with $r_i \in [i]$ defines a bijection $\sigma \in \mathfrak{S}$ iff for every i the nondecreasing sequence $r_{i,i}, r_{i,i+1}, \dots$ defined recursively by (2) is bounded. In this case σ is given by (3).*

Now let $\alpha = (\alpha_k)$ be a strictly increasing sequence of positive integers. For notational convenience we shall assume the sequence infinite, but the considerations to follow also apply to finite sequences with obvious modifications. We require that

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty. \quad (4)$$

Let $P^{(\alpha,1)}$ be the product measure on $[1] \times [2] \times \dots$ which makes the coordinates r_i independent and satisfying

- (i) $r_i \equiv i$ for $i \notin \{\alpha_1 + 1, \alpha_2 + 1, \dots\}$,
- (ii) r_i is uniformly distributed on $[i - 1]$ for $i \in \{\alpha_1 + 1, \alpha_2 + 1, \dots\}$.

Lemma 2.2. *A random sequence (r_i) with distribution $P^{(\alpha,1)}$ determines, via (2) and (3), a random element of \mathfrak{S} almost surely.*

Proof. Fix i and condition on the event $r_i = s$. If $i \notin \{\alpha_1, \alpha_2, \dots\}$ then $r_{i,i+1} = s$. If $i \in \{\alpha_1, \alpha_2, \dots\}$ then the expected value of $r_{i,i+1}$ is $s + s/i$. Iterating we see that the expected value of $r_{i,j}$ converges, as $j \rightarrow \infty$, to

$$s \prod_{k: \alpha_k \geq i} \left(1 + \frac{1}{\alpha_k}\right),$$

which is finite in view of (4). Therefore, Fatou's lemma implies that $r_{i,j}$ is bounded in j and Lemma 2.1 can be applied. \square

Using the correspondence between rank sequences (r_i) and infinite permutations we shall consider $P^{(\alpha,1)}$ as a measure on \mathcal{O} supported by \mathfrak{S} . Similarly to \mathfrak{S}_n , for $\sigma \in \mathfrak{S}$ we define position j to be a record if $\sigma(j) = \max_{i \in [j]} \sigma(i)$. Then under $P^{(\alpha,1)}$ the records are positions not of the kind $\alpha_k + 1$.

The dual algorithm There is a dual stochastic algorithm that produces $P^{(\alpha,1)}$ via the entries of the inverse infinite permutation $\sigma^{-1}(1), \sigma^{-1}(2), \dots$. Note that position of integer 1 belongs to $\{1, \alpha_1 + 1, \alpha_2 + 1, \dots\}$, and that in terms of ranks it is $\sigma^{-1}(1) = \max\{i : r_i = 1\}$. Hence introducing a random

variable ν_1 with distribution

$$\text{Prob}\{\nu_1 = 0\} = \prod_{m=1}^{\infty} \left(1 - \frac{1}{\alpha_m}\right), \quad (5)$$

$$\text{Prob}\{\nu_1 = k\} = \frac{1}{\alpha_k} \prod_{m=1}^{\infty} \left(1 - \frac{1}{\alpha_{k+m}}\right), \quad k = 1, 2, \dots \quad (6)$$

the position of 1 can be defined as $\sigma^{-1}(1) = y_1$, where

$$y_1 = 1(\nu_1 = 0) + (\alpha_{\nu_1} + 1)1(\nu_1 \neq 0).$$

Given the value $\sigma^{-1}(1)$, define a new sequence α' by the following rules

- (i) if $\sigma^{-1}(1) = 1$ and $\alpha_1 \geq 2$ then $\alpha'_k = \alpha_k - 1$ for $k \geq 1$,
- (ii) if $\sigma^{-1}(1) = 1$ and $\alpha_1 = 1$ then $\alpha'_k = \alpha_{k+1} - 1$ for $k \geq 1$,
- (iii) if $\sigma^{-1}(1) = \alpha_i + 1$ then $\alpha'_k = \alpha_k$ for $k < i$ and $\alpha'_k = \alpha_{k+1} - 1$ for $k \geq i$.

Let ν_2 have distribution as in the right-hand side of (5), (6) but with α' in place of α . Finally, let $\sigma^{-1}(2)$ be the y_2 th element of $\mathbb{N} \setminus \{\sigma^{-1}(1)\}$ for $y_2 = 1(\nu_2 = 0) + (\alpha_{\nu_2} + 1)1(\nu_2 \neq 0)$. Then we iterate on $\mathbb{N} \setminus \{\sigma^{-1}(1), \sigma^{-1}(2)\}$ and so on.

The algorithm can be taken as an alternative definition of $P^{(\alpha,1)}$. In Section 5 we will give a more direct proof that all positions are eventually filled, hence the algorithm indeed yields a random permutation $\sigma^{-1} \in \mathfrak{S}$.

Finally, we construct a larger family of RD-measures by breeding P^* with $P^{(\alpha,1)}$'s. Fix α satisfying (4) and $0 < p \leq 1$. Split \mathbb{N} in two infinite subsets N_1 and N_2 by assigning each integer independently to N_1 with probability p and to N_2 with probability $1 - p$. Using increasing bijections we can identify N_1 and N_2 with two copies of \mathbb{N} . The order is constructed by requiring that every $i \in N_1$ precedes every $j \in N_2$ and, using the identifications, by ordering N_1 according to $P^{(\alpha,1)}$ and ordering N_2 according to P^* . The resulting measure is denoted $P^{(\alpha,p)}$.

Let Ω denote the set comprised of point $*$ and of pairs (α, p) , where α is a strictly increasing sequence of elements of $\mathbb{Z}_+ \cup \{+\infty\}$ satisfying (4), and $0 < p \leq 1$. Note that α is either infinite sequence of integers, or there finitely many integers and all further elements of α are $+\infty$. The space Ω is a topological cone obtained by collapsing one face of the cylinder $\{\alpha\} \times [0, 1]$ in the point $*$. In this topology convergence to (α, p) is component-wise, and converges to $*$ means that the p -component goes to 0.

Theorem 2.3. *Measures $P^{(\alpha,p)}$ with $(\alpha,p) \in \Omega$ and P^* comprise the set $\text{ext } \mathcal{M}_R(\mathcal{O})$ of extreme RD-measures. The topology on Ω agrees with the topology of weak convergence of measures on \mathcal{O} .*

Denote ω the generic point of Ω (that is, either $*$ or some (α,p)) and let P^ω be the corresponding measure. A general convex analysis fact known as Choquet theorem (see e.g. [11]) says that Theorem 2.3 implies the following statement.

Corollary 2.4. *For every $P \in \mathcal{M}_R(\mathcal{O})$ there exists a unique probability measure μ on Ω , such that P is representable as a convex mixture with weights μ of the extreme record-dependent measures, i.e.*

$$P = \int_{\Omega} P^\omega \mu(d\omega).$$

The next result is a law of large numbers for the extremes.

Proposition 2.5. *Let O be a random element of \mathcal{O} distributed according to $P^{(\alpha,p)}$. Then*

(i) *for every k the k th non-record position in $\pi_n^\infty(O)$ converges to $\alpha_k + 1$ almost surely, as $n \rightarrow \infty$,*

(ii)

$$\liminf_{n \rightarrow \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} = p \quad \text{a.s.}$$

Remark 1. If $\alpha_k = +\infty$, then under convergence to $\alpha_k + 1$ we mean that if for large n the k th non-record position in $\pi_n^\infty(O)$ exists, then it converges to $+\infty$.

Remark 2. Under $P^{(\alpha,p)}$ the number of records is asymptotically linear in n , so that $|R(\pi_n^\infty(O))|/(np) \rightarrow 1$ in probability. Under P_n^* position of n has uniform distribution on $[n]$, hence relation (ii) holds with $p = 0$.

In the next sections we prove Theorem 2.3 and Proposition 2.5.

3 The branching graph representation

In this section we recast the setting of RD-measures on permutations within a general formalism of central measures on branching graphs (see e.g. [16], [18]).

The succession of permutations of different sizes and their record sets is representable in the form of an infinite graded graph \mathcal{R} . It is convenient to

encode each admissible $\rho \in [n]$ into a binary word $\rho(1) \dots \rho(n)$ starting with $\rho(1) = 1$. For instance, $\{1, 3, 4\} \subset [5]$ becomes 10110. Let \mathcal{R}_n denote the set of all 2^{n-1} such binary words of length n .

Consider a graded graph \mathcal{R} with the set of vertices $\bigcup_{n=1}^{\infty} \mathcal{R}_n$, and with edges connecting some vertices on neighboring levels according to the rule: two vertices $\rho = \rho(1) \dots \rho(n) \in \mathcal{R}_n$ and $\tau = \tau(1) \dots \tau(n+1) \in \mathcal{R}_{n+1}$ are connected by an edge, denoted $\rho \nearrow \tau$, if there exists $k \in [n+1]$ such that

- (i) $\rho(i) = \tau(i)$ for $i < k$,
- (ii) $\tau(k) = 1$,
- (iii) $\tau(i) = 0$ for $i > k$.

The first four levels of \mathcal{R} are shown in Figure 1.

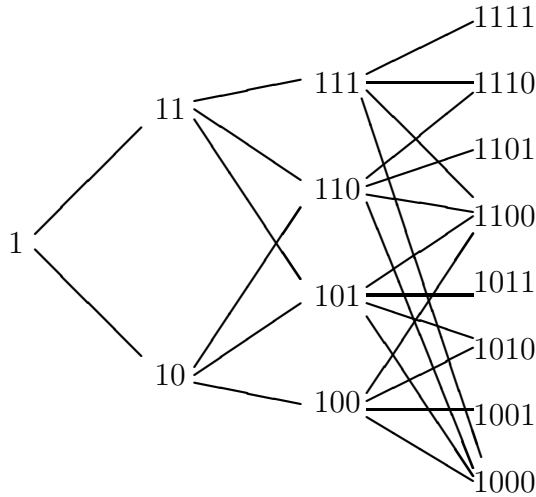


Figure 1: The first four levels of graph \mathcal{R} .

A (standard) path in \mathcal{R} is a sequence of vertices (ρ_i) such that $\rho_i \in \mathcal{R}_i$ and $\rho_i \nearrow \rho_{i+1}$. Let Γ be the set of infinite paths $\rho_1 \nearrow \rho_2 \nearrow \dots$, and let Γ_n be the set of paths $\rho_1 \nearrow \dots \nearrow \rho_n$ of length n . We view Γ as a projective limit of finite sets Γ_n and we equip Γ with the usual topology of projective limit of discrete spaces. Recall that $R(\sigma) \in \mathcal{R}_n$ for $\sigma \in \mathfrak{S}_n$.

Proposition 3.1. *The map*

$$\Phi_n : \sigma \rightarrow (R(\pi_j^n(\sigma)), j \in [n])$$

(where π_n^n is the identity map) is a bijection between \mathfrak{S}_n and Γ_n . Similarly, the map

$$\Phi : \mathcal{O} \rightarrow (R(\pi_n^\infty(\mathcal{O})), n \in \mathbb{N})$$

is a homeomorphism between \mathcal{O} and Γ .

Proof. Let $\sigma_n = \pi_n^\infty(\mathcal{O})$. As in the proof of Lemma 1.2, $\sigma_n^{-1}(n)$ is uniquely determined by $R(\sigma_n)$ and $R(\sigma_{n-1})$. On the other hand, σ_n is uniquely determined by $\sigma_j^{-1}(j)$, $j \in [n]$, by the virtue of a correspondence analogous to the bijection between \mathfrak{S}_n and the sequence of n ranks. \square

Identifying finite paths with permutations, and infinite paths with orders on \mathbb{N} we shall use the same symbols as above for measures and projections. For instance π_n^∞ will denote the projection $\Gamma \rightarrow \Gamma_n$ which cuts the tail of a path up to the first n terms.

A probability measure P_n on Γ_n is called *central* if probability of a path $\rho_1 \nearrow \rho_2 \cdots \nearrow \rho_n$, depends only on ρ_n , i.e. all paths with fixed endpoint ρ_n are equally likely. Similarly, a probability measure P on Γ is central if all its projections $P_n = \hat{\pi}_n^\infty(P)$ on Γ_n are central. Let $\mathcal{M}_C(\Gamma)$ denote the space of all central measures on Γ . We remark that each $P \in \mathcal{M}_C(\Gamma)$ is associated with a random walk which moves along the paths in \mathcal{R} and has some standard reverse transition probabilities determined by the condition of centrality, see e.g. [16].

The following statement is straightforward from Proposition 3.1.

Proposition 3.2. *The map $\hat{\Phi}$ is a an affine isomorphism of the convex sets $\mathcal{M}_R(\mathcal{O})$ and $\mathcal{M}_C(\Gamma)$.*

For $\rho_n \in \mathcal{R}_n$, the *elementary* measure P^{ρ_n} is the central measure on Γ_n supported by the set of paths of length n with endpoint ρ_n . By the correspondence between \mathfrak{S}_n and Γ_n , the elementary measure corresponds to the uniform distribution on the set of permutations $\sigma \in \mathfrak{S}_n$ with fixed records $R(\sigma) = \rho_n$. The next standard fact (which was reproved many times by many authors, see e.g. [22, Proposition 10.8], [5, Theorem 1.1] and references therein) is the main technical tool to determine the set $\mathcal{M}_C(\Gamma)$.

Lemma 3.3. *Let P be an extreme point of the convex set $\mathcal{M}_C(\Gamma)$. Then for P -almost all paths $\rho_1 \nearrow \rho_2 \nearrow \dots$ in \mathcal{R}*

$$\lim_{n \rightarrow \infty} \hat{\pi}_k^n(P^{\rho_n})(A) \rightarrow \hat{\pi}_k^\infty(P)(A), \quad (7)$$

for all $A \subset \Gamma_k$ and $k \in \mathbb{N}$.

The family of probability measures $P \in \mathcal{M}_C(\mathcal{R})$ representable, like in (7), as limits of elementary measures along some path (ρ_n) will be called *Martin boundary* of the graph \mathcal{R} . (Sometimes the Martin boundary is understood in a wider sense as the set of limits along arbitrary sequence (ρ_n) .) By Lemma 3.3 the set of extremes $\text{ext } \mathcal{M}_C(\Gamma)$ is a part of the Martin boundary. Convergence (7) is the same as the weak convergence of projections on every Γ_k . With this in mind we shall simply write $P^{\rho_n} \rightarrow P$. The boundary problem is straightforwardly re-formulated in terms of permutations.

4 The Martin boundary identification

Theorem 4.1. *Let $(\rho_n) \in \Gamma$ be a path such that the elementary RD-measures on \mathfrak{S}_n weakly converge, i.e. $P^{\rho_n} \rightarrow P$. Then $P = P^\omega$ for some $\omega \in \Omega$. Thus, the Martin boundary of \mathcal{R} can be identified with Ω .*

To prove the result we need some auxiliary propositions.

For $\rho \in \mathcal{R}_n$ with k zeroes let $\ell_1(\rho) < \ell_2(\rho) < \dots < \ell_k(\rho)$ be the positions of zeros listed in increasing order,

$$\{\ell_1(\rho), \dots, \ell_k(\rho)\} = \{i : \rho(i) = 0\},$$

and define

$$L(\rho) := \prod_{i=1}^k \left(1 - \frac{1}{\ell_i(\rho) - 1}\right).$$

The following algorithm produces a random permutation with distribution P^ρ . Let $m_1 > m_2 > \dots > m_{n-k} = 1$ be the positions of 1's listed in the decreasing order. Since $\sigma^{-1}(n)$ is the maximum record we must have $\sigma(m_1) = n$. Next, $\sigma(m_1 + 1), \dots, \sigma(n)$ is a equiprobable sample from $\{1, \dots, n-1\} = \{1, \dots, n\} \setminus \{\sigma(m_1)\}$. Furthermore, $\sigma(m_2)$ is the maximum element of $\{1, \dots, n\} \setminus \{\sigma(m_1), \dots, \sigma(n)\}$, thus there is only one choice for $\sigma(m_2)$ after $\sigma(m_1), \dots, \sigma(n)$ have been determined. Now $\sigma(m_3 + 1), \dots, \sigma(m_2 - 1)$ is a equiprobable sample from $\{1, \dots, n\} \setminus \{\sigma(m_1), \dots, \sigma(n)\} \setminus \{\sigma(m_2)\}$. The process is continued until all positions are filled.

The algorithm for sampling permutations from P^ρ readily implies:

Proposition 4.2. *Let $\rho \in \mathcal{R}_n$ and let σ be a random permutation from \mathfrak{S}_n distributed according to P^ρ . The random variable $\sigma^{-1}(1)$ has the following*

distribution:

$$P^\rho(\sigma^{-1}(1) = h) = \begin{cases} \prod_{i=1}^k \left(1 - \frac{1}{\ell_i - 1}\right), & \text{if } h = 1, \\ \frac{1}{\ell_j - 1} \prod_{i=j+1}^k \left(1 - \frac{1}{\ell_i - 1}\right), & \text{if } h = \ell_j, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.3. *Let $\rho \in \mathcal{R}_n$, and let σ be a random permutation from \mathfrak{S}_n distributed according to P^ρ . For $2 \leq t \leq n$ the conditional distribution of $\sigma^{-1}(t)$ given $\sigma^{-1}(1) = s_1, \dots, \sigma^{-1}(t-1) = s_{t-1}$ is*

$$P^\rho(\sigma^{-1}(t) = h \mid \sigma^{-1}(1) = s_1, \dots, \sigma^{-1}(t-1) = s_{t-1}) = \begin{cases} \prod_{i=1}^{k'} \left(1 - \frac{1}{\ell'_i - 1 - w(\ell'_i)}\right), & \text{if } h = \min(\{1, \dots, n\} \setminus \{s_1, \dots, s_{t-1}\}), \\ \frac{1}{\ell'_j - 1 - w(\ell'_j)} \prod_{i=j+1}^{k'} \left(1 - \frac{1}{\ell'_i - 1 - w(\ell'_i)}\right), & \text{if } h = \ell'_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $\ell'_1 < \dots < \ell'_{k'}$ satisfy

$$\{\ell'_1, \dots, \ell'_{k'}\} = \{\ell_1, \dots, \ell_k\} \setminus \{s_1, \dots, s_{t-1}\}$$

and

$$w(x) = \left| \{s_1, \dots, s_{t-1}\} \cap \{1, \dots, x-1\} \right|.$$

Proposition 4.4. *If a path $(\rho_n) \in \Gamma$ satisfies $L(\rho_n) \rightarrow 0$, then $P^{\rho_n} \rightarrow P^*$.*

Proof. Observe that if for $\sigma \in \mathfrak{S}_n$ positions of $1, \dots, m$ are not records, i.e. $\{\sigma^{-1}(1), \dots, \sigma^{-1}(m)\} \subset [n] \setminus R(\sigma)$ then the set of records remains unaltered if the positions of $1, \dots, m$ are exchanged. Therefore, under the RD-measure P^{ρ_n} the permutation $\pi_m^n(\sigma)$ is uniformly distributed given $\{\sigma^{-1}(1), \dots, \sigma^{-1}(m)\} \subset [n] \setminus R(\sigma)$. Finally, by Propositions 4.2 and 4.3 if $L(\rho_n) \rightarrow 0$ then

$$P^{\rho_n}(\sigma^{-1}(m) \notin R(\sigma)) \rightarrow 1$$

for every m . □

Proposition 4.5. *Suppose $P^{\rho_n} \rightarrow P$ with $P \neq P^*$ for some path $(\rho_n) \in \Gamma$. Then there exists a 0 – 1 sequence $\rho_\infty = (\rho_\infty(1), \rho_\infty(2), \dots)$ such that*

$$\lim_{n \rightarrow \infty} \rho_n(i) = \rho_\infty(i)$$

for every i .

Proof. Suppose that for some i the sequence $\rho_n(i)$ does not converge. It means that for infinitely many n_j we have $\rho_{n_j}(i) = 0$ and $\rho_{n_j-1}(i) = 1$. Then, since $\rho_{n_j-1} \nearrow \rho_{n_j}$, we have

$$\rho_{n_j}(i) = \rho_{n_j}(i+1) = \dots = \rho_{n_j}(n_j) = 0.$$

Therefore, $L(\rho_{n_j}) \rightarrow 0$ as $j \rightarrow \infty$ and Proposition 4.4 implies that $\widehat{\pi}_k^{n_j}(P^{\rho_{n_j}})$ converges to the uniform measure on \mathfrak{S}_k , so $P^{\rho_{n_j}} \rightarrow P^*$ which is a contradiction. \square

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. If $L(\rho_n) \rightarrow 0$ the limit is P^* . Otherwise,

1. $L(\rho_n)$ is bounded away from zero,
2. there exists a $0-1$ sequence $\rho_\infty = (\rho_\infty(1), \rho_\infty(2), \dots)$ such that $\lim_{n \rightarrow \infty} \rho_n(i) = \rho_\infty(i)$.

by Propositions 4.4 and 4.5.

Passing, if necessary, to a subsequence we assume that $L(\rho_n)$ converges to a number $0 < p_1 \leq 1$ as $n \rightarrow \infty$. Now let $\ell_1(\rho_\infty) < \ell_2(\rho_\infty) < \dots$ be positions of zeros in ρ_∞ :

$$\{\ell_1(\rho_\infty), \ell_2(\rho_\infty), \dots\} = \{i : \rho_\infty(i) = 0\}.$$

Set $\alpha_i = \ell_i(\rho_\infty) - 1$ for all i such that $\ell_i(\rho_\infty)$ is defined and set $\alpha_i = +\infty$ for all greater i . Observe that convergence of $L(\rho_n)$ implies that $\prod (1 - 1/\alpha_i)$ converges to a number p_2 and $p_1 \leq p_2 \leq 1$. Now set $p = p_1/p_2$. We claim that

$$P_k = \widehat{\pi}_k^\infty(P^{(\alpha, p)}).$$

The claim becomes straightforward when we compare the algorithmic description of $P^{(\alpha, p)}$ (i.e. the dual algorithm given in Section 2) and the description of the elementary measures P^ρ given in Propositions 4.2 and 4.3. \square

Corollary 4.6. *The measures P^ω , $\omega \in \Omega$ are record-dependent.*

Proof. Indeed, by Theorem 4.1 they are weak limits of record-dependent measures. \square

5 Laws of large numbers

In this section we exploit the algorithmic description of measures $P^{(\alpha,p)}$ to prove, in particular, Proposition 2.5 and finish the proof of Theorem 2.3.

First, suppose that $p = 1$ and fix a sequence α such that $\sum_{i=1}^{\infty} 1/\alpha_i < \infty$. Recall, that the dual algorithm for $P^{(\alpha,1)}$ constructs successively the entries $\sigma^{-1}(1), \sigma^{-1}(2), \dots$ of the inverse permutation $\sigma^{-1} : \mathbb{N} \rightarrow \mathbb{N}$.

Lemma 5.1. *For every $\varepsilon > 0$ there exist constants $C > 1$ and n_0 such that the estimate*

$$P^{(\alpha,1)}(\sigma^{-1}(k) > Cn \mid \sigma^{-1}(1) = s_1, \dots, \sigma^{-1}(k-1) = s_{k-1}) < \varepsilon. \quad (8)$$

holds for $n > n_0$, $k \leq n$ and arbitrary distinct s_1, \dots, s_{k-1} .

Proof. For shorthand, we write Q for the conditional probability in (8). As follows from the description of the dual algorithm in Section 2,

$$P^{(\alpha,1)}(\sigma^{-1}(1) > Cn) = 1 - \prod_{i:\alpha_i > Cn} \left(1 - \frac{1}{\alpha_i}\right).$$

More generally, a similar formula holds for Q with α_i being replaced with other sequence β_i . As follows from the description of the procedure of obtaining this new sequence β_i given in Section 2, to get β_i , we, first, pass from α_i to some subsequence and then, perhaps, subtract from each term some integral numbers which are less than k . Therefore,

$$Q \leq 1 - \prod_{i:\alpha_i > Cn} \left(1 - \frac{1}{\alpha_i - k}\right) \leq 1 - \prod_{i:\alpha_i > Cn} \left(1 - \frac{1}{\alpha_i - n}\right).$$

Since $\ln(1+x) \geq 2x$ for $-1/2 \leq x \leq 0$, we have the following estimate

$$\begin{aligned} -\frac{1}{2} \ln \left(\prod_{i:\alpha_i > Cn} \left(1 - \frac{1}{\alpha_i - n}\right) \right) &\leq \sum_{i:\alpha_i > Cn} \frac{1}{\alpha_i - n} \\ &= \sum_{i:\alpha_i > Cn} \frac{1}{\alpha_i} + \sum_{i:\alpha_i > Cn} \frac{n}{\alpha_i(\alpha_i - n)} \leq \sum_{i:\alpha_i > Cn} \frac{1}{\alpha_i} + \sum_{j=Cn}^{\infty} \frac{n}{(j-n-1)(j-n)} \\ &= \sum_{i:\alpha_i > Cn} \frac{1}{\alpha_i} + \frac{n}{Cn} = \sum_{i:\alpha_i > Cn} \frac{1}{\alpha_i} + \frac{1}{C} \end{aligned}$$

Now choose small $\delta > 0$ such that $1 - e^{-\delta} < \varepsilon$. Let $C > \frac{1}{4\delta}$ and choose n_0 such that

$$\sum_{i:\alpha_i > Cn_0} \frac{1}{\alpha_i} < \delta/4$$

(this is possible, since $\sum 1/\alpha_i$ converges). Then for $n > n_0$ we obtain $Q < 1 - e^{-\delta} < \varepsilon$ as wanted. \square

Proposition 5.2. *Let O be the random order with distribution $P^{(\alpha,1)}$ and let $\sigma_n = \pi_n^\infty$ be the projection of O on \mathfrak{S}_n . Then*

$$\frac{\sigma_n^{-1}(n)}{n} \rightarrow 1$$

$P^{(\alpha,1)}$ -almost surely.

Proof. Choose $\varepsilon > 0$. Recall that a real-valued random variable X stochastically dominates another such variable Y if for any bounded non-decreasing function f we have $Ef(X) \geq Ef(Y)$. Observe that Lemma 5.1 implies that the random variable $X = |\{1 \leq i \leq n \mid \sigma^{-1}(i) \leq Cn\}|$ stochastically dominates the sum of $n - n_0$ independent Bernoulli random variables with the probability of 1 equal to $1 - \varepsilon$ (see Lemma 1.1 in and [20] and Lemma 1 in [23]). Now using a standard large deviations estimate (see e.g. [15], Chapter 27) for the sum of independent Bernoulli random variables we conclude that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$P^{(\alpha,1)}(|\{1 \leq i \leq n \mid \sigma^{-1}(i) \leq Cn\}| > (1 - 2\varepsilon)n) > 1 - \exp(-C_2n) \quad (9)$$

for $n > n_1$.

Observe that the set $\{\sigma^{-1}(1), \dots, \sigma^{-1}(n)\}$ has the following structure: it is the union of some interval $\{1, \dots, M\}$ and a subset of the set $\{\alpha_1 + 1, \alpha_2 + 1, \dots\}$.

The convergence of the series $\sum 1/\alpha_i$ implies that

$$\frac{|\{i \mid \alpha_i \leq Cn\}|}{n} \rightarrow 0.$$

Therefore, (9) yields that

$$P^{(\alpha,1)}(\{1, \dots, \lfloor (1 - 3\varepsilon)n \rfloor\} \subset \{\sigma^{-1}(1), \dots, \sigma^{-1}(n)\}) > 1 - \exp(-C_2n)$$

for $n > n_2$. But on the event $\{1, \dots, \lfloor (1 - 3\varepsilon)n \rfloor\} \subset \{\sigma^{-1}(1), \dots, \sigma^{-1}(n)\}$ we have $(\pi_{n+1}^\infty(O))^{-1}(n+1) > (1 - 3\varepsilon)n$. Hence, for $n > n_2$ we have

$$P^{(\alpha,1)}\left(\frac{\sigma_{n+1}^{-1}(n+1)}{n} > (1 - 3\varepsilon)\right) > 1 - \exp(-C_2n). \quad (10)$$

Since the series

$$\sum_{n=n_2+1}^{\infty} \exp(-C_2n)$$

converges, the estimate (10) and the Borel-Cantelli lemma imply that almost surely for all but finitely many n we have

$$\frac{\sigma_{n+1}^{-1}(n+1)}{n} > (1 - 3\varepsilon).$$

Consequently, almost surely

$$\liminf_{n \rightarrow \infty} \frac{\sigma_n^{-1}(n)}{n} > 1 - 3\varepsilon.$$

To finish the proof it remains to observe that $\varepsilon > 0$ is arbitrary and $\sigma_n^{-1}(n) \leq n$ always holds. \square

Corollary 5.3. *The dual algorithm for $P^{(\alpha,1)}$ eventually fills every position, so that the output is indeed a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$.*

Proof. Indeed, in the proof of Proposition 5.2 we have shown that for every k the probability of the event $\{1, \dots, k\} \subset \{\sigma^{-1}(1), \dots, \sigma^{-1}(n)\}$ tends to 1 as $n \rightarrow \infty$. \square

Now we seek for analogue of Proposition 5.2 for more general P^ω .

Proposition 5.4. *Let O be the random order with distribution P^* and let $\sigma_n = \pi_n^\infty$ be the projection of O on \mathfrak{S}_n . Then*

$$\liminf_{n \rightarrow \infty} \frac{\sigma_n^{-1}(n)}{n} \rightarrow 0.$$

P^ -almost surely.*

Proof. Under P^* permutation σ_{n-1} and position $\sigma_n^{-1}(n)$ are independent, and the latter is uniformly distributed on $[n]$. Since $P_n^*(\sigma_n^{-1}(n) = 1) = 1/n$, the event $\{\sigma_n^{-1}(n) = 1\}$ almost surely occurs infinitely often as $n \rightarrow \infty$, and the statement is obvious. \square

The statement for general $P^{(\alpha,p)}$ is an interpolation between Propositions 5.2 and 5.4.

Proposition 5.5. *Let $0 < p < 1$, then for the order O with distribution $P^{(\alpha,p)}$*

$$\liminf_{n \rightarrow \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} \rightarrow p.$$

$P^{(\alpha,p)}$ -almost surely.

Proof. Let O_0 and O_α be two independent linear orders on \mathbb{N} such that the distributions of O_0 and O_α are P^* and $P^{(\alpha,1)}$, respectively. Recall that the $P^{(\alpha,p)}$ -distributed order O is constructed from O_0 and O_α : we divide \mathbb{N} into two subsets \mathbb{N}_1 and \mathbb{N}_2 using a sequence of independent Bernoulli random variables with parameter p . Then we view O_0 as a linear order on \mathbb{N}_2 , view O_α as a linear order on \mathbb{N}_1 and require that \mathbb{N}_1 precedes \mathbb{N}_2 .

We see that $\sigma = \pi_n^\infty(O)$ is constructed as follows. Let $M_1 = \mathbb{N}_1 \cap [n]$ and $M_2 = \mathbb{N}_2 \cap [n]$. The permutation $\sigma_1 = \pi_{|M_1|}^\infty(O_\alpha)$ uniquely defines a permutation $\bar{\sigma}_1$ of the set M_1 and $\sigma_2 = \pi_{|M_2|}^\infty(O_0)$ uniquely defines a permutation $\bar{\sigma}_2$ of the set M_2 . Permutation σ is obtained by first writing $\bar{\sigma}_1$ and then writing $\bar{\sigma}_2$.

Let us analyze $\sigma^{-1}(n)$. Choose $\varepsilon > 0$. Almost surely for large enough n we have

1. $p - \varepsilon \leq |M_1|/n \leq p + \varepsilon$,
2. $1 - \varepsilon \leq (\sigma_1)^{-1}(|M_1|)/|M_1| \leq 1$.

The latter is just the statement of Proposition 5.2 and the former follows from the law of large numbers for Bernoulli scheme. Now if $n \in M_1$, then $\sigma^{-1}(n) = \sigma_1^{-1}(|M_1|)$ and, thus,

$$\frac{\sigma^{-1}(n)}{n} \geq (p - \varepsilon)(1 - \varepsilon).$$

If $n \in M_2$, then

$$\sigma^{-1}(n) > |M_1| \geq (p - \varepsilon)n.$$

Since ε is arbitrary, we conclude that

$$\liminf_{n \rightarrow \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} \geq p.$$

Next, using Proposition 5.4 we conclude that almost surely there exists an increasing sequence n_m such that for $n = n_m$, $m = 1, 2, \dots$ we have

1. $n \in M_2$,
2. $\sigma_2^{-1}(|M_2|) = 1$.

This implies that for large enough m ,

$$\sigma^{-1}(n) = |M_1| + 1 \leq (p + \varepsilon)n + 1.$$

Therefore,

$$\liminf_{n \rightarrow \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} \leq (p + \varepsilon)$$

Since ε is arbitrary, we are done. □

Proposition 5.6. *If $p > 0$, then under $P^{(\alpha,p)}$ the position of the i th non-record in $\pi_k^\infty(O)$ converges to $\alpha_i + 1$ as $k \rightarrow \infty$ almost surely.*

Proof. First, suppose that $p = 1$ and recall the algorithmic description of $P^{(\alpha,1)}$. Note that the permutation $\pi_k^\infty(O)$ is read from the order of numbers $1, \dots, k$ after first k steps of the algorithm. Moreover, observe that if after k steps of the algorithm all positions $1, \dots, \alpha_i + 1$ are filled, then $\alpha_i + 1$ is precisely the position of i th non-record in $\pi_k^\infty(O)$. Therefore, our claim is implied by the Corollary 5.3.

In case of general p note that additional numbers are situated at the end of the permutation and, thus, do not affect positions of first non-records. \square

Proof of Theorem 2.3. The set of extremes $\text{ext}\mathcal{M}_R(\mathcal{O})$ is contained in the Martin boundary by Lemma 3.3. On the other hand, by Proposition 2.5 each measure $P^\omega, \omega \in \Omega$, satisfies a law of large numbers which is specific for this particular P^ω . It follows that the supports of P^ω 's are disjoint, hence none of them can be represented as a nontrivial convex mixture over the Martin boundary. Thus every P^ω is extreme, so $\text{ext}\mathcal{M}_R(\mathcal{O}) = \{P^\omega, \omega \in \Omega\}$. The coincidence of topologies immediately follows from the explicit description of measures P^ω given in Section 2. \square

6 Two connections

Order-invariant measures on causal sets We describe now a connection of record-dependent measures $P^{(\alpha,1)}$ to the recent work on random partial orders [2, 3].

A partial order \triangleleft on \mathbb{N} defines a *causal set* $(\mathbb{N}, \triangleleft)$ if every element is preceded by finitely many other elements. A *natural extension* of \triangleleft is a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ which is order-preserving, i.e. $i \triangleleft j$ implies $\sigma^{-1}(i) < \sigma^{-1}(j)$. A *stem* is a finite collection of positions j_1, \dots, j_k such that there exists a natural extension with $\sigma^{-1}(1) = j_1, \dots, \sigma^{-1}(k) = j_k$.

If j_1, \dots, j_k is a stem, every $D_\ell = \{j_1, \dots, j_\ell\}$, $1 \leq \ell \leq k$, is a down-set (or lower ideal). A stem can be identified with a chain of down-sets $D_1 \subset \dots \subset D_k$ where $|D_\ell| = \ell$. It is not hard to see that infinite chain of down-sets $D_1 \subset D_2 \subset \dots$ (where $|D_\ell| = \ell$) with $\cup D_\ell = \mathbb{N}$ uniquely corresponds to a natural extension of \triangleleft .

Brightwell and Luczak [2, 3] introduced the notion of *order-invariant measure*, which is a probability measure P on the set of natural extensions of \triangleleft such that

$$P(\sigma^{-1}(1) = j_1, \dots, \sigma^{-1}(k) = j_k) = P(\sigma^{-1}(1) = \ell_1, \dots, \sigma^{-1}(k) = \ell_k),$$

provided $\{j_1, \dots, j_k\} = \{\ell_1, \dots, \ell_k\}$. The condition means that probability of a stem only depends on the corresponding down-set $D_k = \{j_1, \dots, j_k\}$. It is possible to interpret order-invariant measures as central measures on the path space of the graded graph of down-sets.

Let (α_k) be a strictly increasing sequence of integers as in Section 3, and let (β_k) be the (infinite) sequence complimentary to $(\alpha_k + 1)$, so that $\{\alpha_1 + 1, \alpha_2 + 1, \dots\} \cup \{\beta_1, \beta_2, \dots\} = \mathbb{N}$. Consider a partial order \triangleleft generated by the relations

$$\beta_1 \triangleleft \beta_2 \triangleleft \dots, \quad \alpha_i + 1 \triangleleft \max\{\beta_k : \beta_k \leq \alpha_i\},$$

which means that (β_k) is a chain, and each segment $\beta_k + 1, \beta_k + 2, \dots, \beta_{k+1} - 1$ is an antichain covered by β_k .

Obviously from the definitions, $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ is a natural extension of \triangleleft if and only if $\{\beta_k\}$ is the set of records of σ .

Proposition 6.1. *$P^{(\alpha,1)}$ is the unique order-invariant measure for the causal set $(\mathbb{N}, \triangleleft)$.*

Sketch of the proof. Every finite down-set with elements arranged by increase is a sequence γ of the kind

$$1, 2, \dots, \beta_k, \beta_k + 1, \beta_k + 2, \dots, \beta_{k+1} - 1, \alpha_{i_1} + 1, \dots, \alpha_{i_\ell} + 1,$$

where either of the segments $\beta_k + 1, \beta_k + 2, \dots, \beta_{k+1} - 1$ or $\alpha_{i_1}, \dots, \alpha_{i_\ell}$ can be empty. Call such γ admissible.

The conditions ensuring that the set of records is (β_k) and that γ is admissible impose constraints on permutation that can be expressed in terms of the r_i . We illustrate this with γ of the form

$$1, \dots, a, b, c$$

where $a + 1 \in \{\beta_k\}$ and $b, c \in \{\alpha_k + 1\}$. The constraints on the ranks become $r_j = j$ for $j \notin \{\alpha_k + 1\}$, $r_j < j$ for $j \in \{\alpha_k + 1\}$ and, to guarantee the admissibility,

$$\begin{aligned} r_i &\geq a + 1 \quad \text{for } a < i < b, \\ r_b &\leq a + 1, \\ r_i &\geq a + 2 \quad \text{for } b < i < c, \\ r_c &= a + 2, \\ r_i &\geq a + 3 \quad \text{for } i > c. \end{aligned}$$

Under $P^{(\alpha,1)}$ the r_i 's are independent and each r_i is uniformly distributed on a suitable range. Therefore each possible stem associated with γ has

the same probability, equal to the probability of admissible realization of $r_j, j \leq \alpha_{i_\ell}$. The order-invariance of the measure follows.

For *finite* causal set $([n], \triangleleft)$ the analog of order-invariant measure is the uniform distribution on the extensions of \triangleleft . The uniqueness assertion follows from the fact that $P^{(\alpha,1)}$ is a weak limit of such measures as $n \rightarrow \infty$ along (β_i) , and condition (4) ensures that the limit is a bijection. We omit details, see [3] (Section 9) for a more general result. \square

The Young-Fibonacci lattice The *Young-Fibonacci graph (lattice)* was introduced by Stanley [26] and Fomin [6]. They found out that it shares lots of the features with the Young graph, which is the object naturally arising in representation theory and combinatorics. In particular, Stanley proved that both graphs are *differential posets*.

The vertices of the Young-Fibonacci graph at level n are labeled by words in the alphabet $\{1, 2\}$, with the sum of digits equal n . For instance 1111, 211, 121, 112, 22 are words on level $n = 4$. The number of vertices on n th level is the n th Fibonacci number. Successors of a word are obtained by either inserting a 1 in any position within the rightmost contiguous block of 2's, or by replacing the rightmost 1 with 2. For instance, 2212 has followers 12212, 21212, 22112, 2222.

Goodman and Kerov [12] studied the Markov boundary of the Young-Fibonacci graph. Comparing with their result, it is seen that the Martin boundary of the Young-Fibonacci graph has the same conical structure as Ω . The apex is the *Plancherel measure* which like our P^* appears as a push-forward of the uniform distribution on permutations. The base is a discrete space comprised of the measures which like our $P^{(\alpha,1)}$'s are parameterized by infinite words in the alphabet $\{1, 2\}$ with 'rare' occurrences of 2's, to satisfy a condition similar to (4). The Plancherel measure of the Young-Fibonacci graph was further studied in [8].

The reasonings of [12] are very much different from the present paper. Goodman and Kerov intensively use the relation to a certain non-commutative algebra introduced by Okada [21]. Note also that unlike the Young-Fibonacci graph, the graph of record-sets \mathcal{R} is not a differential poset. Thus, it seems that no direct bijection between (measures on) \mathcal{R} and Young-Fibonacci graph can exist. This makes the coincidence of the Markov boundaries for these graphs even more intriguing. The authors would be glad to find any good explanation for this fact.

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